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A new "dual" symmetry structure of the KP hierarchy

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Abstract

A new infinite set of commuting additional ("ghost") symmetries is proposed for the KP-type integrable hierarchy. These symmetries allow for a Lax representation in which they are realized as standard isospectral flows. This gives rise to a new double-KP hierarchy embedding "ghost" and original KP-type Lax hierarchies connected to each other via a "duality" mapping exchanging the isospectral and "ghost" "times". A new representation of the 2D Toda lattice hierarchy as a special Darboux–Bäcklund orbit of the double-KP hierarchy is found and parametrized entirely in terms of (adjoint) eigenfunctions of the original KP subsystem. © 1998 Published by Elsevier Science B.V.

1. Introduction

Additional non-isospectral symmetries [1,2] of Kadomtsev–Petviashvili (KP) type hierarchies of integrable nonlinear evolution equations play a prominent rôle, especially, in view of the connection between integrable systems and non-perturbative string theory (see Ref. [3] and references therein). Fixed points of subalgebras of additional symmetries, realized in the context of (multi-)matrix models of string theory via the so-called Virasoro and *W*-constraints, allow us to express the tau-functions of the underlying integrable hierarchies as partition functions of statistical mechanical models of random matrices.

Besides the above principal applications, there exist other interesting aspects of additional symmetries of integrable hierarchies which deserve further detailed study. The purpose of the present Letter is to study *Abelian* subalgebras of additional symmetries, often called also "ghost" symmetries. The most succinct formulation of additional symmetries in KP-type hierarchies is given in Refs. [2] in the language of the algebra of pseudo-differential operators which is the formalism we are employing in the present Letter. Since, by construction, commuting "ghost" symmetries commute also with the isospectral flows, it is natural to consider them as additional KP "times". More precisely, we are interested in *infinite* sets of commuting "ghost" symmetries. Then, the natural idea arises to reformulate the infinite number of commuting "ghost" symmetry flows as ordinary isospectral flows of some other "ghost" KP integrable hierarchy. In this way the original isospectral flows of the initial KP system themselves will play the rôle of additional commuting "ghost" symmetries of the second "ghost" KP hierarchy. Thus we obtain a new double-KP integrable system embedding both the original

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and the “ghost” KP hierarchies in such a way that the latter appear “dual” to each other under interchanging of isospectral and “ghost”-flow “times”. It is this property of “duality” which is the most characteristic feature of the present approach. As a first application we present a simple construction of the two-dimensional Toda lattice hierarchy as a special non-standard Darboux–Bäcklund orbit of the new double-KP continuum hierarchy.

Let us stress the following point. Infinite sets of commuting additional (“ghost”) symmetries, extending the original KP hierarchy, have also been considered in Refs. [4]. The principal difference between the above approaches and the present construction is as follows. In Ref. [4] the Lax operators are always given in terms of pseudo-differential operators acting in one and the same space, i.e., pseudo-differential operators with respect to $x \equiv t_1^{(1)}$ (the first evolution parameter of the first subsystem of KP “times”). In other words, one subsystem of evolution parameters plays in Ref. [4] a distinguished rôle. Unlike this, in the present approach both the isospectral KP flows of the initial KP hierarchy as well as its “ghost” symmetry flows appear totally symmetric within the full double-KP system. Both types of flows are formulated in terms of Lax operators acting in two *different* spaces: pseudo-differential operators with respect to $x \equiv t_1$ (the first evolution parameter of the initial KP system) and $\bar{x} \equiv \bar{t}_1$ (the first “ghost” symmetry flow, which becomes the first evolution parameter of the second “ghost” KP system), respectively. Moreover, our construction manifestly exhibits the pertinent “duality” between the two ordinary KP subsystems within the double-KP hierarchy which is lacking in Ref. [4].

2. Background on KP hierarchy and “ghost” symmetries

In what follows we use the Sato formalism of pseudo-differential operator calculus (see, e.g., Ref. [5]) to describe KP-type integrable hierarchies. The main object is the pseudo-differential Lax operator $\mathcal{L}$ obeying an infinite set of evolution equations[1] with respect to the KP “times” $(t) \equiv (t_1 \equiv x, t_2, \ldots)$,

$$\mathcal{L} = D + \sum_{i=1}^{\infty} u_i D^{-i} , \qquad \frac{\partial \mathcal{L}}{\partial t_l} = [\, (\mathcal{L}^l)_+ \, , \, \mathcal{L} \,] , \qquad l = 1,2,\ldots . \tag{1}$$

Equivalently, one can represent (1) in terms of the dressing operator W whose pseudo-differential series are directly expressed in terms of the so-called tau-function $\tau(t)$,

$$\mathcal{L} = W D W^{-1} , \qquad \frac{\partial W}{\partial t_l} = -(\mathcal{L}^l)_- W , \qquad W = \sum_{n=0}^{\infty} \frac{p_l(-[\partial])\tau(t)}{\tau(t)} D^{-l} , \tag{2}$$

with the notation $[y] \equiv (y_1, y_2/2, y_3/3, \ldots)$ for any multi-variable $(y) \equiv (y_1, y_2, y_3, \ldots)$ and with $p_k(t)$ being the Schur polynomials, $\exp \sum_{l\geqslant 1} \lambda^l t_l = \sum_{k \geqslant 0} \lambda^k p_k(t)$. The tau-function is related to the Lax operator as

$$\partial_x \frac{\partial}{\partial t_l} \ln \tau(t) = \operatorname{Res} \mathcal{L}^l . \tag{3}$$

In the present approach a crucial notion is that of (adjoint) eigenfunctions ((adjoint) eigenfunctions) $\Phi(t)$, $\Psi(t)$ of the KP hierarchy satisfying

$$\frac{\partial \Phi}{\partial t_k} = \mathcal{L}^k_+(\Phi) , \qquad \frac{\partial \Psi}{\partial t_k} = -(\mathcal{L}^*)^k_+(\Psi) . \tag{4}$$

[1] We shall employ the following notations: for any (pseudo-)differential operator A and a function f, the symbol $A(f)$ will indicate action of A on f, whereas the symbol Af will denote just the operator product of A with the zero-order (multiplication) operator f. The symbol D stands for the differential operator $\partial/\partial x$, whereas $\partial \equiv \partial_x$ will denote the derivative on a function. Further, in what follows the subscripts $(\pm)$ of any pseudo-differential operator $A = \sum_j a_j D^j$ denote its purely differential part $(A_+ = \sum_{j\geqslant 0} a_j D^j)$ or its purely pseudo-differential part $(A_- = \sum_{j \geqslant 1} a_{-j} D^{-j})$, respectively.

The (adjoint) Baker–Akhiezer (BA) "wave" functions $\psi_{BA}(t,\lambda) = W(\exp(\xi(t,\lambda)))$ and $\psi^*_{BA}(t,\lambda) = (W^*)^{-1}(\exp(-\xi(t,\lambda)))$ (with $\xi(t,\lambda) \equiv \sum_{l=1}^{\infty} t_l \lambda^l$) are (adjoint) eigenfunctions which, in addition, also satisfy the spectral equations of the form $\mathcal{L}^{(*)}(\psi^{(*)}_{BA}(t,\lambda)) = \lambda \psi^{(*)}_{BA}(t,\lambda)$. As shown in Ref. [6], any (adjoint) eigenfunction possesses a spectral representation of the form

$$\Phi(t) = \int \delta\lambda\, \varphi(\lambda)\psi_{BA}(t,\lambda)\,, \qquad \Psi(t) = \int \delta\lambda\, \varphi^*(\lambda)\psi^*_{BA}(t,\lambda)\,, \tag{5}$$

with spectral "densities" $\varphi^{(*)}(\lambda)$ given by

$$\varphi(\lambda) = \frac{1}{\lambda}\psi^*_{BA}(t',\lambda)\Phi(t' + [\lambda^{-1}])\,, \qquad \varphi^*(\lambda) = \frac{1}{\lambda}\psi_{BA}(t',\lambda)\Psi(t' - [\lambda^{-1}])\,, \tag{6}$$

where the multi-time $t' = (t'_1, t'_2, \ldots)$ is taken at some arbitrary fixed value (as shown in Ref. [6], the spectral integrals (5) themselves do *not* depend on the choice of t').

The spectral densities (6) can also be expressed [6] in terms of the so-called *squared eigenfunction potential* $S(\Phi,\Psi)$ [7] which yields a well-defined unique expression for the inverse derivative ∂_x^{-1} of a product of arbitrary pair of eigenfunction and adjoint eigenfunction [6],

$$\partial^{-1}(\Phi(t)\Psi(t)) \equiv S(\Phi,\Psi) = -\int\int \delta\lambda\,\delta\mu\,\varphi^*(\lambda)\varphi(\mu)\frac{e^{\xi(t,\mu)-\xi(t,\lambda)}}{\lambda-\mu}\,\frac{e^{\sum_1^{\infty}(1/l)(\lambda^{-l}-\mu^{-l})\partial/\partial t_l}\tau(t)}{\tau(t)}\,. \tag{7}$$

This will always be the case for all instances of appearance of inverse derivatives in the sequel.

Finally, let us recall the basic facts about "ghost" symmetries of the generic KP hierarchy. A "ghost" symmetry is defined through an action of a vector field $\hat{\partial}_\alpha$ on the KP Lax operator or the dressing operator [8],

$$\hat{\partial}_\alpha \mathcal{L} = [\,\mathcal{M}_\alpha\,,\,\mathcal{L}\,]\,, \qquad \hat{\partial}_\alpha W = \mathcal{M}_\alpha W\,, \qquad \mathcal{M}_\alpha \equiv \sum_{a\in\{\alpha\}} \Phi_a D^{-1}\Psi_a\,, \tag{8}$$

where $(\Phi_a,\Psi_a)_{a\in\{\alpha\}}$ are some set of functions indexed by $\{\alpha\}$. Commutativity of $\hat{\partial}_\alpha$ with $\partial/\partial t_l$ implies that $(\Phi_a,\Psi_a)_{a\in\{\alpha\}}$ is a set of pairs of (adjoint) eigenfunctions of $\mathcal{L}$.

Now, for the general (adjoint) eigenfunctions Φ,Ψ of $\mathcal{L}$ we define new generalized "ghost" symmetry flows,

$$\hat{\partial}_\alpha \Phi = \sum_{a\in\{\alpha\}} \Phi_a \partial^{-1}(\Psi_a\Phi) - \mathcal{F}^{(\alpha)}\,, \qquad \hat{\partial}_\alpha \Psi = \sum_{a\in\{\alpha\}} \Psi_a \partial^{-1}(\Phi_a\Psi) + \mathcal{F}^{*(\alpha)}\,. \tag{9}$$

Note the additional inhomogeneous terms $\mathcal{F}^{(\alpha)}$, $\mathcal{F}^{*(\alpha)}$ which themselves are (adjoint) eigenfunctions of $\mathcal{L}$ (1) and which are absent in the traditional approach of Refs. [8,7] (see, however, Ref. [9], where inhomogeneous terms of the type in Eq. (9) have been considered in the particular case of constrained KP hierarchies). It is crucial for what follows that their presence is in general allowed by requirements of *commutativity of two different "ghost" flows* $\hat{\partial}_\alpha$ and $\hat{\partial}_\beta$ and the integrability condition $\left[\,\hat{\partial}_\alpha - \mathcal{M}_\alpha\,,\,\hat{\partial}_\beta - \mathcal{M}_\beta\,\right] = 0$ following from the definition (8).

3. Construction of the "ghost"-flow KP hierarchy

Consider an infinite system of independent (adjoint) eigenfunctions $\{\Phi_j,\Psi_j\}_{j=1}^{\infty}$ of $\mathcal{L}$ and define the following infinite set of the "ghost" symmetry flows,

$$\frac{\partial}{\partial \bar{t}_s}\mathcal{L} = [\,\mathcal{M}_s\,,\,\mathcal{L}\,]\,, \qquad \mathcal{M}_s = \sum_{j=1}^{s} \Phi_{s-j+1} D^{-1}\Psi_j\,, \tag{10}$$

$$\frac{\partial}{\partial \bar{t}_s}\Phi_k = \sum_{j=1}^{s} \Phi_{s-j+1}\partial^{-1}(\Psi_j\Phi_k) - \Phi_{k+s}\,, \qquad \frac{\partial}{\partial \bar{t}_s}\Psi_k = \sum_{j=1}^{s} \Psi_j\partial^{-1}(\Phi_{s-j+1}\Psi_k) + \Psi_{k+s}\,, \tag{11}$$

$$\frac{\partial}{\partial \bar{t}_s}F = \sum_{j=1}^{s} \Phi_{s-j+1}\partial^{-1}(\Psi_j F)\,, \qquad \frac{\partial}{\partial \bar{t}_s}F^* = \sum_{j=1}^{s} \Psi_j\partial^{-1}(\Phi_{s-j+1}F^*)\,, \tag{12}$$

where $s, k = 1, 2, \ldots$, and (F^*) F denote generic (adjoint) eigenfunctions which do not belong to the "ghost" symmetry generating set $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$. With the choice of the inhomogeneous terms as in (11) it is easy to show that the "ghost" symmetry flows $\partial/\partial\bar{t}_s$ do indeed commute, i.e., the ∂-pseudo-differential operators $\mathcal{M}_s$ (10) satisfy

$$\frac{\partial}{\partial \bar{t}_s}\mathcal{M}_r - \frac{\partial}{\partial \bar{t}_r}\mathcal{M}_s - [\,\mathcal{M}_s\,,\,\mathcal{M}_r\,] = 0\,. \tag{13}$$

In particular, for the first "ghost" symmetry flow $\partial/\partial\bar{t}_1 \equiv \bar{\partial}$, we have

$$\bar{\partial}\Phi_k = \Phi_1\partial^{-1}(\Psi_1\Phi_k) - \Phi_{k+1}\,, \qquad \bar{\partial}\Psi_k = \Psi_1\partial^{-1}(\Phi_1\Psi_k) + \Psi_{k+1}\,, \qquad \bar{\partial}F = \Phi_1\partial^{-1}(\Psi_1 F)\,. \tag{14}$$

Eqs. (14), in turn, imply the following 2D Toda lattice (2DTL) like equations for the Wronskians of Φ_j's and Ψ_j's, respectively,

$$\partial\bar{\partial}\ln W_k = \Phi_1\Psi_1 - \frac{W_{k+1}W_{k-1}}{{W_k}^2}\,, \qquad \partial\bar{\partial}\ln \mathcal{W}_k = \Phi_1\Psi_1 + \frac{\mathcal{W}_{k+1}\mathcal{W}_{k-1}}{{\mathcal{W}_k}^2}\,. \tag{15}$$

Here and below, use will be made of the following short-hand notations for the Wronskian-type determinants,

$$W_k \equiv W_k\,[\Phi_1, \ldots, \Phi_k] = \det \|\partial^{\alpha-1}\Phi_\beta\|\,, \qquad \alpha, \beta = 1, \ldots, k\,, \qquad \mathcal{W}_k \equiv W_k\,[\Psi_1, \ldots, \Psi_k]\,, \tag{16}$$

$$W_k(F) \equiv W_{k+1}\,[\Phi_1, \ldots, \Phi_k, F]\,, \qquad \mathcal{W}_k(F^*) \equiv W_{k+1}\left[\Psi_1, \ldots, \Psi_k, F^*\right]\,, \tag{17}$$

$$\widetilde{W}\left[f_1, \ldots, f_{k+1}; f\right] \equiv \det_{k+1} \left\| \begin{matrix} \partial^{\alpha-1}f_\beta & \partial^{\alpha-1}f_{k+1} \\ \partial^{-1}(f_\beta f) & \partial^{-1}(f_{k+1} f) \end{matrix} \right\|\,. \tag{18}$$

Consider now the τ-function of $\mathcal{L}$ and let us act with $\partial/\partial\bar{t}_s$ on both sides of (3), obtaining

$$\frac{\partial}{\partial \bar{t}_s}\ln\tau = -\sum_{j=1}^{s}\partial^{-1}(\Phi_{s-j+1}\Psi_j)\,, \tag{19}$$

using (10) as well as the t_r-flow equations $(\partial/\partial t_r)\mathcal{M}_s = [\,\mathcal{L}_+^r\,,\,\mathcal{M}_s\,]_-$. In particular, for $s = 1$, Eq. (19) together with (14) yields

$$\bar{\partial}\ln\tau = -\partial^{-1}(\Phi_1\Psi_1)\,, \qquad \bar{\partial}\ln(\Phi_1\tau) = -\frac{\Phi_2}{\Phi_1}\,, \qquad \bar{\partial}\ln(\Psi_1\tau) = \frac{\Psi_2}{\Psi_1}\,. \tag{20}$$

Taking into account the first equation (20), we can rewrite (15) in the standard 2DTL form,

$$\partial\bar{\partial}\ln(W_k\tau) = -\frac{(W_{k+1}\tau)(W_{k-1}\tau)}{(W_k\tau)^2}\,, \qquad \partial\bar{\partial}\ln(\mathcal{W}_k\tau) = \frac{(\mathcal{W}_{k+1}\tau)(\mathcal{W}_{k-1}\tau)}{(\mathcal{W}_k\tau)^2}\,. \tag{21}$$

Using the last equations of (14) and (15), we can reexpress the action of the ∂-pseudo-differential operators $\mathcal{M}_s$ (10) on eigenfunctions as *ordinary* $\bar{\partial}$*-differential* operators.

Lemma 1. For any generic eigenfunction F of $\mathcal{L}$, which does not appear within the set $\{\Phi_j\}$ in (10) (and whose "ghost" symmetry flows are given by last equation of (12)), we have

$$\frac{\partial}{\partial \bar{t}_s}(F/\Phi_1) = \bar{M}_s(F/\Phi_1)\,, \tag{22}$$

$$\bar{M}_s \equiv \sum_{j=1}^{s}\left(\sum_{k=j}^{s}\frac{\Phi_{s-k+1}}{\Phi_1}\frac{\mathcal{W}_{j-1}(\Psi_k)}{\mathcal{W}_j}\right)\left(\bar{D}-\bar{\partial}\ln\frac{\mathcal{W}_{j-1}}{\mathcal{W}_{j-2}}\right)\cdots(\bar{D}-\bar{\partial}\ln\Psi_1)\left(\bar{D}-\bar{\partial}\ln\frac{1}{\Phi_1}\right)-\frac{\partial}{\partial\bar{t}_s}\ln\Phi_1\,, \tag{23}$$

where the $\bar{\partial}$-differential operators $\bar{M}_s$ satisfy the standard form of Zakharov–Shabat (ZS) "zero-curvature" equations with respect to the $\bar{t}_s$-flows,

$$\frac{\partial}{\partial\bar{t}_s}\bar{M}_r - \frac{\partial}{\partial\bar{t}_r}\bar{M}_s - [\,\bar{M}_s\,,\,\bar{M}_r\,] = 0\,. \tag{24}$$

Eq. (24) is a consequence of (13).

According to Ref. [10], for any ZS system (as in (24)) there always exists a unique triangular coordinate transformation in the space of evolution parameters such that the (transformed) ZS differential operators acquire the standard KP form, i.e., $\bar{M}_s = (\bar{\mathcal{L}}^s)_+$ for some KP-type $\bar{\partial}$-*pseudo-differential* operator $\bar{\mathcal{L}}$. It turns out that the "ghost" ZS operators (23) have already the right form.

Proposition 1. The $\bar{\partial}$-differential ZS operators $\bar{M}_s$ (23) can be expressed, using the short-hand notations (16), as

$$\bar{M}_s = (\bar{\mathcal{L}}^s)_+\,, \qquad \bar{\mathcal{L}} = \bar{D} + \sum_{k=1}^{\infty} b_k\left(\bar{D}+\bar{\partial}\ln\frac{W_{k+1}}{W_k}\right)^{-1}\cdots\left(\bar{D}+\bar{\partial}\ln\frac{W_2}{\Phi_1}\right)^{-1}, \tag{25}$$

$$b_1 = -\bar{\partial}(\Phi_2/\Phi_1)\,,$$

$$b_k = -\bar{\partial}(\Phi_{k+1}/\Phi_1) + \sum_m \frac{P_m^{(k)}(\bar{\partial}^{\alpha}(\Phi_l/\Phi_1)\,;\,\bar{\partial}(W_s/W_{s-1}))}{Q_m^{(k)}(W_s/W_{s-1})} \qquad \text{for } k = 2,3,\ldots\,, \tag{26}$$

where $P_m^{(k)}, Q_m^{(k)}$ denote monomials with respect to the indicated arguments with $\alpha \geqslant 0$, $2 \leqslant l \leqslant k$, $1 \leqslant s \leqslant k$.

Eqs. (25) and (22) imply

Corollary 1. For any generic eigenfunction F of the initial $\mathcal{L}$ (1), which does not appear within the "ghost"-flow generating set $\{\Phi_j\}$ in (10), the function $\bar{F} \equiv F/\Phi_1$ is automatically an eigenfunction of the "ghost" Lax operator $\bar{\mathcal{L}}$ (25).

Remark. The pseudo-differential series of the original Lax operator $\mathcal{L}$ (1) can always be rearranged into a form similar to (25),

$$\mathcal{L} = D + \sum_{k=1}^{\infty} a_k\left(D-\partial\ln\frac{\mathcal{W}_{k+1}}{\mathcal{W}_k}\right)^{-1}\cdots\left(D-\partial\ln\frac{\mathcal{W}_2}{\Psi_1}\right)^{-1}, \tag{27}$$

with appropriate coefficients a_k. Expressions (27), (25) are very suggestive when discussing Darboux–Bäcklund (DB) orbits and the connection to the 2DTL to which we now turn our attention.

4. Darboux–Bäcklund orbits

Let us first introduce the following *non-standard* orbit of successive DB transformations for the original KP system,

$$\mathcal{L}(n+1) = T_1(n)\mathcal{L}(n)T_1^{-1}(n)\,, \qquad T_1(n) = \Phi_1 D \Phi_1^{-1} \equiv \Phi_1^{(n)} D \Phi_1^{(n)\,-1}\,, \tag{28}$$

$$\Phi_l^{(n+1)} = \Phi_1^{(n)}\partial\left(\frac{\Phi_{l+1}^{(n)}}{\Phi_1^{(n)}}\right), \qquad l \geqslant 1\,,$$

$$\Psi_1^{(n+1)} = \frac{1}{\Phi_1^{(n)}}\,, \qquad \Psi_j^{(n+1)} = -\frac{1}{\Phi_1^{(n)}}\partial^{-1}\left(\Phi_1^{(n)}\Psi_{j-1}^{(n)}\right), \qquad j \geqslant 2\,, \tag{29}$$

for transformations in "positive" direction, as well as adjoint DB transformations, i.e., transformations in "negative" direction,

$$\mathcal{L}(n-1) = \hat{T}_1(n)\mathcal{L}(n)\hat{T}_1^{-1}(n)\,, \qquad \hat{T}_1(n) = \Psi_1 D \Psi_1^{-1} \equiv \Psi_1^{(n)} D \Psi_1^{(n)\,-1}\,, \tag{30}$$

$$\Phi_1^{(n-1)} = \frac{1}{\Psi_1^{(n)}}\,, \qquad \Phi_l^{(n-1)} = \frac{1}{\Psi_1^{(n)}}\partial^{-1}\left(\Psi_1^{(n)}\Phi_{l-1}^{(n)}\right), \qquad l \geqslant 2\,,$$

$$\Psi_j^{(n-1)} = -\Psi_1^{(n)}\partial\left(\frac{\Psi_{j+1}^{(n)}}{\Psi_1^{(n)}}\right), \qquad j \geqslant 1\,. \tag{31}$$

In what follows, the DB "site" index (n) on (adjoint) eigenfunctions will be skipped for brevity whenever this would not lead to ambiguities.

Remark. Let us stress the non-canonical form of the (adjoint) DB transformations (29), (31) on the "ghost" symmetry generating (adjoint) eigenfunctions. On the other hand, for a generic eigenfunction F the (adjoint) DB transformations read as usual [7],

$$F^{(n+1)} = \Phi_1^{(n)}\partial\left(\frac{F^{(n)}}{\Phi_1^{(n)}}\right) = \left(\partial - \partial \ln \Phi_1^{(n)}\right)F^{(n)}\,, \qquad F^{(n-1)} = \frac{1}{\Psi_1^{(n)}}\partial^{-1}\left(\Psi_1^{(n)}F^{(n)}\right). \tag{32}$$

We now obtain the following important proposition.

Proposition 2. "Ghost" symmetries (10) commute with DB transformations (28)–(31), i.e., the "ghost" symmetry generators (10) $\mathcal{M}_s \equiv \mathcal{M}_s(n) = \sum_{j=1}^{s} \Phi_{s-j+1}^{(n)} D^{-1} \Psi_j^{(n)}$ transform on the DB-orbit as

$$\mathcal{M}_s(n) \longrightarrow \mathcal{M}_s(n\pm 1) = \overset{(\wedge)}{T}(n)\mathcal{M}_s(n)\overset{(\wedge)}{T}{}^{-1}(n) + \left(\frac{\partial}{\partial \bar{t}_s}\overset{(\wedge)}{T}(n)\right)\overset{(\wedge)}{T}{}^{-1}(n)$$

$$= \sum_{j=1}^{s} \Phi_{s-j+1}^{(n\pm 1)} D^{-1} \Psi_j^{(n\pm 1)}\,. \tag{33}$$

The "ghost" KP Lax operator (25) transforms, accordingly, as

$$\bar{\mathcal{L}}(n+1) = \left(\frac{1}{\Phi_1^{(n+1)}}\bar{D}^{-1}\Phi_1^{(n+1)}\right)\bar{\mathcal{L}}(n)\left(\frac{1}{\Phi_1^{(n+1)}}\bar{D}\Phi_1^{(n+1)}\right). \tag{34}$$

Iterating the (adjoint) DB transformations (29), (31), we obtain

$$\Phi_s^{(n+k)} = \frac{W_{k+1}[\Phi_1^{(n)},\dots,\Phi_k^{(n)},\Phi_{k+s}^{(n)}]}{W_k[\Phi_1^{(n)},\dots,\Phi_k^{(n)}]}, \quad k \geqslant 1$$

$$\Psi_s^{(n+k)} = -\frac{\widetilde{W}_k[\Phi_1^{(n)},\dots,\Phi_k^{(n)};\Psi_{s-k}^{(n)}]}{W_k[\Phi_1^{(n)},\dots,\Phi_k^{(n)}]}, \qquad 1 \leqslant k \leqslant s-1,$$

$$\Psi_s^{(n+k)} = (-1)^{s-1}\frac{W_{k-1}[\Phi_1^{(n)},\dots,\Phi_{k-s}^{(n)},\Phi_{k-s+2}^{(n)},\dots,\Phi_k^{(n)}]}{W_k[\Phi_1^{(n)},\dots,\Phi_k^{(n)}]}, \qquad k \geqslant s, \tag{35}$$

$$\Phi_s^{(n-k)} = (-1)^{k-1}\frac{\widetilde{W}_k[\Psi_1^{(n)},\dots,\Psi_k^{(n)};\Phi_{s-k}^{(n)}]}{W_k[\Psi_1^{(n)},\dots,\Psi_k^{(n)}]}, \qquad 1 \leqslant k \leqslant s-1,$$

$$\Phi_s^{(n-k)} = (-1)^{s-k}\frac{W_{k-1}[\Psi_1^{(n)},\dots,\Psi_{k-s}^{(n)},\Psi_{k-s+2}^{(n)},\dots,\Psi_k^{(n)}]}{W_k[\Psi_1^{(n)},\dots,\Psi_k^{(n)}]}, \qquad k \geqslant s,$$

$$\Psi_s^{(n-k)} = (-1)^{k}\frac{W_{k+1}[\Psi_1^{(n)},\dots,\Psi_k^{(n)},\Psi_{k+s}^{(n)}]}{W_k[\Psi_1^{(n)},\dots,\Psi_k^{(n)}]}, \qquad k \geqslant 1, \tag{36}$$

where $s \geqslant 1$, and we have used notations (16)–(18) for the Wronskian(-like) determinants.

5. Double-KP hierarchy

We now are able to introduce the double-KP hierarchy and its tau-functions. We first construct an infinite set of (adjoint) eigenfunctions $(\bar{\Phi}_j,\bar{\Psi}_j)_{j=1}^{\infty}$ for the "ghost" Lax operator $\bar{\mathcal{L}}$ (25) in terms of the initial set of (adjoint) eigenfunctions $(\Phi_j,\Psi_j)_{j=1}^{\infty}$ of $\mathcal{L}$ defining the "ghost" symmetry flows (10)–(12). Taking F = const in (22), we find that $\bar{\Phi}_1^{(n)} \equiv 1/\Phi_1^{(n)} = \Psi_1^{(n+1)}$ is an eigenfunction of $\bar{\mathcal{L}}(n)$ for any "site" n on the DB-orbit (28)–(31). Therefore, taking into account (34), we deduce that $\bar{\Psi}_1^{(n-1)} \equiv 1/\bar{\Phi}_1^{(n)} = \Phi_1^{(n)}$ is an adjoint eigenfunction of $\bar{\mathcal{L}}(n-1)$ again for any DB "site" n. The rest of the (adjoint) eigenfunctions $\bar{\Phi}_j,\bar{\Psi}_j$ for $\bar{\mathcal{L}}$ $(j \geqslant 2)$ is constructed in such a way that their DB-orbit will have the following form consistent with the DB-orbit of $\bar{\mathcal{L}}$ (34),

$$\bar{\Phi}_j^{(n-1)} = \bar{\Phi}_1^{(n)}\bar{\partial}\left(\frac{\bar{\Phi}_{j+1}^{(n)}}{\bar{\Phi}_1^{(n)}}\right), \qquad j \geqslant 1,$$

$$\bar{\Psi}_1^{(n-1)} = \frac{1}{\bar{\Phi}_1^{(n)}}, \qquad \bar{\Psi}_l^{(n-1)} = -\frac{1}{\bar{\Phi}_1^{(n)}}\bar{\partial}^{-1}(\bar{\Phi}_1^{(n)}\bar{\Psi}_{l-1}^{(n)}), \qquad l \geqslant 2, \tag{37}$$

$$\bar{\Phi}_1^{(n+1)} = \frac{1}{\bar{\Psi}_1^{(n)}}, \qquad \bar{\Phi}_l^{(n+1)} = \frac{1}{\bar{\Psi}_1^{(n)}}\bar{\partial}^{-1}(\bar{\Psi}_1^{(n)}\bar{\Phi}_{l-1}^{(n)}), \qquad l \geqslant 2,$$

$$\bar{\Psi}_j^{(n+1)} = -\bar{\Psi}_1^{(n)}\bar{\partial}\left(\frac{\bar{\Psi}_{j+1}^{(n)}}{\bar{\Psi}_1^{(n)}}\right), \qquad j \geqslant 1, \tag{38}$$

$$\bar{F}^{(n-1)} = \bar{\Phi}_1^{(n)}\bar{\partial}\left(\frac{\bar{F}^{(n)}}{\bar{\Phi}_1^{(n)}}\right) = (\bar{\partial} - \bar{\partial}\ln\bar{\Phi}_1^{(n)})\bar{F}^{(n)}, \tag{39}$$

where $\bar{F}$ is a generic eigenfunction of $\bar{\mathcal{L}}$.

The explicit form of $\{\bar{\Phi}_j, \bar{\Psi}_j\}$ reads using notations (16), (17),

$$\bar{\Phi}_l^{(n)} \equiv \bar{\Phi}_l = (-1)^l \frac{1}{\Phi_1} \bar{\partial}^{-1} \Phi_1 \Psi_1 \bar{\partial}^{-1} \frac{\mathcal{W}_2}{\Psi_1^2} \bar{\partial}^{-1} \frac{\mathcal{W}_3 \Psi_1}{(\mathcal{W}_2)^2} \bar{\partial}^{-1} \cdots \bar{\partial}^{-1} \frac{\mathcal{W}_{l-1}\mathcal{W}_{l-3}}{(\mathcal{W}_{l-2})^2}, \qquad l \geqslant 2, \tag{40}$$

$$\bar{\Phi}_1^{(n)} \equiv \bar{\Phi}_1 = \frac{1}{\Phi_1}, \qquad \bar{\Psi}_j^{(n)} \equiv \bar{\Psi}_j = (-1)^{j-1} \frac{W_2}{\Phi_1} \bar{\partial}^{-1} \frac{W_3 \Phi_1}{(W_2)^2} \bar{\partial}^{-1} \cdots \bar{\partial}^{-1} \frac{W_{j+1} W_{j-1}}{(W_j)^2}, \qquad j \geqslant 1. \tag{41}$$

Remark. For later use let us explicitly write down the k-step iteration of DB transformations on $\Phi_1^{(n)}, \bar{\Phi}_1^{(n)}$,

$$\Phi_1^{(n-k)} = (-1)^{k-1} \frac{\mathcal{W}_{k-1}}{\mathcal{W}_k}, \qquad \bar{\Phi}_1^{(n+k)} = \frac{1}{\Phi_1^{(n+k)}} = \frac{W_{k-1}}{W_k}. \tag{42}$$

Collecting the above results, we obtain:

Proposition 3. Both Lax operators, the initial $\mathcal{L}$ (27) and the "ghost" one $\bar{\mathcal{L}}$ (25), define a double-KP integrable system,

$$\frac{\partial}{\partial t_r}\mathcal{L} = [\,(\mathcal{L}^r)_+, \mathcal{L}\,], \qquad \frac{\partial}{\partial \bar{t}_s}\mathcal{L} = [\,\mathcal{M}_s, \mathcal{L}\,],$$
$$\frac{\partial}{\partial \bar{t}_s}\bar{\mathcal{L}} = [\,(\bar{\mathcal{L}}^s)_+, \bar{\mathcal{L}}\,], \qquad \frac{\partial}{\partial t_r}\bar{\mathcal{L}} = [\,\bar{\mathcal{M}}_r, \bar{\mathcal{L}}\,], \tag{43}$$

where $\mathcal{M}_s$ was introduced in (10) and $\bar{\mathcal{M}}_r$ is its "dual" counterpart defined in terms of the $\bar{\mathcal{L}}$ (adjoint) eigenfunctions (40), (41): $\bar{\mathcal{M}}_r = \sum_{i=1}^{r} \bar{\Phi}_{r-i+1} \bar{D}^{-1} \bar{\Psi}_i$. Accordingly, for generic eigenfunctions F, $\bar{F}$ of $\mathcal{L}$ and $\bar{\mathcal{L}}$, respectively, we have

$$\frac{\partial}{\partial t_r} F = (\mathcal{L}^r)_+(F), \qquad \frac{\partial}{\partial \bar{t}_s} F = \mathcal{M}_s(F), \qquad \frac{\partial}{\partial \bar{t}_s}(F/\Phi_1) = (\bar{\mathcal{L}}^s)_+(F/\Phi_1), \tag{44}$$

$$\frac{\partial}{\partial \bar{t}_s} \bar{F} = (\bar{\mathcal{L}}^s)_+(\bar{F}), \qquad \frac{\partial}{\partial t_r} \bar{F} = \bar{\mathcal{M}}_r(\bar{F}), \qquad \frac{\partial}{\partial t_r}(\Phi_1 \bar{F}) = (\mathcal{L}^r)_+(\Phi_1 \bar{F}). \tag{45}$$

Corollary 2. According to Proposition 3 and (10), there exists a duality mapping between the two scalar KP subsystems of (43) defined by $\mathcal{L}$ and $\bar{\mathcal{L}}$, respectively, under the exchange $(t) \leftrightarrow (\bar{t})$, $\Phi_j \leftrightarrow \bar{\Phi}_j$, $\Psi_j \leftrightarrow \bar{\Psi}_j$.

There exists a simple relation between the tau-functions of $\mathcal{L}$ and $\bar{\mathcal{L}}$. Namely, using (11) and (14) in Eq. (23) for $s = 2$ leads to $\bar{M}_2 \equiv \bar{\mathcal{L}}_+^2 = \bar{\partial}^2 - 2\bar{\partial}(\Phi_2/\Phi_1)$, i.e., $\mathrm{Res}_{\bar{\partial}} \bar{\mathcal{L}} = \bar{\partial}^2 \ln \bar{\tau} = -\bar{\partial}(\Phi_2/\Phi_1)$ which, upon comparing with the second equation (20), implies for $\bar{\tau}$ of $\bar{\mathcal{L}}$: $\bar{\partial}^2 \ln \bar{\tau} = \bar{\partial}^2 \ln(\Phi_1 \tau)$. Applying duality (Corollary 2) to the above equation and to the first equation in (20), we find $\partial^2 \ln \bar{\tau} = \partial^2 \ln(\Phi_1 \tau)$ and $\partial\bar{\partial} \ln \bar{\tau} = \partial\bar{\partial} \ln(\Phi_1 \tau)$. The above relations can be generalized to the following proposition.

Proposition 4. The τ-function of $\bar{\partial}$-Lax operator $\bar{\mathcal{L}}$ (25) is expressed in terms of eigenfunctions and the τ-function of the original ∂-Lax operator $\mathcal{L}$ (27) as follows,

$$\bar{\tau}(t, \bar{t}) = \Phi_1(t, \bar{t})\, \tau(t, \bar{t}), \qquad \frac{p_s(-[\bar{\partial}])\bar{\tau}}{\bar{\tau}} = \frac{\Phi_{s+1}}{\Phi_1}, \tag{46}$$

or, introducing the "site" index of the DB-orbit (28)–(31): $\bar{\tau}^{(n)}(t,\bar{t}) = \tau^{(n+1)}(t,\bar{t})$.

Recalling expressions (2) we derive from the second equation (46) a remarkably simple explicit parametrization of the second "ghost" KP subsystem of (43) in terms of eigenfunctions of the first initial KP system.

Corollary 3. The dressing operator $\bar{W}$ for $\bar{\mathcal{L}}$ (25) has the following explicit form,

$$\bar{\mathcal{L}} = \bar{W}\bar{D}\bar{W}^{-1}, \qquad \bar{W} = 1 + \sum_{s=1}^{\infty} \frac{\Phi_{s+1}}{\Phi_1}\bar{D}^{-s}, \tag{47}$$

where $\{\Phi_s\}_{s=1}^{\infty}$ are the "ghost" symmetry generating eigenfunctions (10) of the original Lax operator (1).

Applying duality (Corollary 2) to Eqs. (47), we also get for the initial "dressing" operator W,

$$\mathcal{L} = WDW^{-1}, \qquad W = 1 + \sum_{s=1}^{\infty} \frac{\bar{\Phi}_{s+1}}{\bar{\Phi}_1} D^{-s}, \tag{48}$$

where $\{\bar{\Phi}_s\}_{s=1}^{\infty}$ are the same as in (40). Therefore, we conclude from (47), (48) and (40), (41) that the whole double-KP hierarchy (43) is parametrized entirely in terms of the infinite set $\{\Phi_j, \Psi_j\}_{j=1}^{\infty}$ of "ghost" symmetry generating (adjoint) eigenfunctions (10) considered as functions of both original isospectral flow and "ghost" symmetry flow parameters $(t,\bar{t})$.

Finally, let us also note that the tau-functions of the double KP hierarchy (46) obey the following generalized Fay identities which easily follow by matching the "spectral" representation (5) for any generic eigenfunction F of the initial KP Lax operator (27) with the corresponding "spectral" representation of $\bar{F} = F/\Phi_1$ as eigenfunction of the "ghost" Lax operator (25),

$$\begin{aligned} &\lambda(\mu-\nu)\,\tau(t-[\kappa^{-1}],\bar{t}-[\lambda^{-1}])\,\bar{\tau}(t,\bar{t}-[\mu^{-1}]-[\nu^{-1}]) \\ &\quad + \nu(\lambda-\mu)\tau(t-[\kappa^{-1}],\bar{t}-[\nu^{-1}])\bar{\tau}(t,\bar{t}-[\lambda^{-1}]-[\mu^{-1}]) \\ &\quad + \mu(\nu-\lambda)\tau(t-[\kappa^{-1}],\bar{t}-[\mu^{-1}])\bar{\tau}(t,\bar{t}-[\lambda^{-1}]-[\nu^{-1}]) = 0, \end{aligned} \tag{49}$$

$$\begin{aligned} &\lambda(\mu-\nu)\,\bar{\tau}(t-[\lambda^{-1}],\bar{t}-[\kappa^{-1}])\,\tau(t-[\mu^{-1}]-[\nu^{-1}],\bar{t}) \\ &\quad + \nu(\lambda-\mu)\bar{\tau}(t-[\nu^{-1}],\bar{t}-[\kappa^{-1}])\tau(t-[\lambda^{-1}]-[\mu^{-1}],\bar{t}) \\ &\quad + \mu(\nu-\lambda)\bar{\tau}(t-[\mu^{-1}],\bar{t}-[\kappa^{-1}])\,\tau(t-[\lambda^{-1}]-[\nu^{-1}],\bar{t}) = 0. \end{aligned} \tag{50}$$

6. 2D Toda lattice hierarchy as Darboux–Bäcklund orbit of double-KP hierarchy

We now turn to the construction of the 2DTL as a special DB-orbit of the double-KP system (43). Recalling Eqs. (42) allows us to rewrite Lax operator expressions (27), (25) in the form

$$\mathcal{L} \equiv \mathcal{L}(n) = D + \sum_{k=1}^{\infty} a_k(n)\,(D - \partial \ln \Phi_1^{(n-k)})^{-1} \cdots (D - \partial \ln \Phi_1^{(n-1)})^{-1}, \tag{51}$$

$$\bar{\mathcal{L}} \equiv \bar{\mathcal{L}}(n) = \bar{D} + \sum_{k=1}^{\infty} b_k(n)\,(\bar{D} - \bar{\partial} \ln \bar{\Phi}_1^{(n+k)})^{-1} \cdots (\bar{D} - \bar{\partial} \ln \bar{\Phi}_1^{(n+1)})^{-1}, \tag{52}$$

where we reintroduced the DB "site" index on the non-standard DB-orbit defined in (28)–(31).

Now, taking into account (32) and (39), one can represent the action of $\mathcal{L}(n), \bar{\mathcal{L}}(n)$ (51), (52) on generic eigenfunctions $F^{(n)}$, $\bar{F}^{(n)}$ at any fixed "site" n of the DB-orbit as action of infinite Jacobi-type matrices Q_{nm}, $\bar{Q}_{nm}$ on infinite column vectors $F^{(n)}$ and $\bar{F}^{(n)}$ ($k \geqslant 1$ below),

$$\mathcal{L}(n)(F^{(n)}) = Q_{nm}F^{(m)}\,, \qquad \bar{\mathcal{L}}(n)(\bar{F}^{(n)}) = \left(\frac{1}{\Phi_1^{(n)}}\bar{Q}_{nm}\Phi_1^{(m)}\right)\bar{F}^{(m)}\,, \tag{53}$$

$$Q_{n,n+k} = \delta_{k,1}\,, \qquad Q_{n,n-k} = a_k(n)\,, \qquad Q_{nn} = \partial \ln \Phi_1^{(n)}\,, \tag{54}$$

$$\bar{Q}_{n,n-k} = \delta_{k,1}\frac{\Phi_1^{(n)}}{\Phi_1^{(n-1)}} = \Phi_1\Psi_1\delta_{k,1}\,, \qquad \bar{Q}_{n,n+k} = b_k(n)\frac{\Phi_1^{(n)}}{\Phi_1^{(n+k)}} = b_k\frac{\Phi_1 W_{k-1}}{W_k}\,, \qquad \bar{Q}_{nn} = -\bar{\partial}\ln\Phi_1^{(n)}\,. \tag{55}$$

Using (53), the (pseudo-)differential Lax equations of the double-KP hierarchy (43)–(45) for any fixed DB "site" n can be equivalently represented as discrete Lax equations for the infinite Jacobi-type matrices (54), (55),

$$Q_{nm}\psi_m = \lambda\psi_n\,, \qquad \frac{\partial}{\partial t_r}\psi_n = (Q^r_+)_{nm}\psi_m\,, \qquad \frac{\partial}{\partial \bar{t}_s}\psi_n = -(\bar{Q}^s_-)_{nm}\psi_m\,, \tag{56}$$

$$\frac{\partial}{\partial t_r}Q = [\,(Q^r)_+\,,\,Q\,]\,, \qquad \frac{\partial}{\partial \bar{t}_s}Q = [\,Q\,,\,(\bar{Q}^s)_-\,]\,,$$

$$\frac{\partial}{\partial \bar{t}_s}\bar{Q} = [\,\bar{Q}\,,\,(\bar{Q}^s)_-\,]\,, \qquad \frac{\partial}{\partial t_r}\bar{Q} = [\,(Q^r)_+\,,\,\bar{Q}\,]\,, \tag{57}$$

where we took the BA function as F, i.e., $F^{(n)}(t) = \psi^{(n)}_{BA}(t,\lambda) \equiv \psi_n$, and where the subscripts $(\pm)$ indicate upper+diagonal/lower-triangular part of the corresponding matrices. The above equations are the Lax equations for the 2DTL hierarchy [11] (see also Ref. [12]). Thus, we showed that the structure of the non-standard DB-orbit (28)–(31) of the double-KP hierarchy coincides with that of the 2DTL hierarchy.

7. Outlook

In this Letter we presented our results on a new "ghost" symmetry structure of the KP system giving rise to a duality between two related KP hierarchies embedded into a *double*-KP system. A detailed exposition with complete proofs will appear elsewhere. It will also address a variety of further interesting issues: (a) the relation (embedding into) of the present double-KP hierarchy (43) to multi-component matrix KP hierarchies [11]; (b) the generalization of the present construction with several infinite sets of "ghost" symmetries; (c) the relation to the random multi-matrix models [12]; (d) the supersymmetric generalization and obtaining a consistent supersymmetric 2DTL hierarchy. The first part of task (d) has already been addressed in Ref. [13].

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